Problems based on Module –I (Metric Spaces)

Ex.1 Let d be a metric on X. Determine all constants K such that

(i) kd , (ii) d + k is a metric on X

Ex.2. Show that

$$\left|d(x,y)-d(z,w)\right| \le d(x,z)+d(y,w) \text{ where } x,y,z,w \in (X,d).$$

Ex.3. Find a sequence which converges to 0, but is not in any space ℓ^p where $1 \le p < \infty$.

Ex.4. Find a sequence x which is in ℓ^p with $p > 1$ but $x \notin \ell^1$.

Ex.5. Let (X,d) be a metric space and A,B are any two non empty subsets of X .Is

$$D(A,B) = \inf_{a \in A\ ,\ b \in B} d(a,b)$$

a metric on the power set of X?

Ex.6. Let (X, d) be any metric space. Is $\left(X,\bar{d}\right)$ a Metric space where $\bar{d} = d(x,y)/\left[1+d(x,y)\right]$.

Ex.7. Let (X_1, d_1) and (X_2, d_2) be metric Spaces and $X = X_1 \times X_2$. .Are $\bar{d}$ as defined below A metric on X ?

(i) $\bar{d}(x,y) = d_1(x_1, y_1) + d_2(x_2, y_2)$; (ii) $\bar{d}(x,y) = \sqrt{d_1^{\,2}(x_1, y_1) + d_2^{\,2}(x_2, y_2)}$

(iii) $\bar{d}(x,y) = \max\left[d_1(x_1,y_1), d_2(x_2,y_2)\right]$, where $x = (x_1, x_2), y = (y_1, y_2)$

Ex.8. Show that in a discrete metric space X, every subset is open and closed.

Ex.9. Describe the closure of each of the following Subsets:

(a) The integers on R.

(b) The rational numbers on R.

(c) The complex number with real and imaginary parts as rational in $\mathbb{C}$.

(d) The disk $\{z : |z| < 1\} \subset \mathbb{C}$.

Ex.10.Show that a metric space X is separable if and only if X has a countable subset Y with the property: For every $\epsilon > 0$ and every $x \in X$ there is a $y \in Y$ such that $d(x, y) < \epsilon$.

Ex.11. If (x_n) and (y_n) are Cauchy sequences in a metric space (X, d), show that (a_n), where $a_n = d(x_n, y_n)$ converges.

Ex.12. Let $a, b \in R$ and $a < b$. Show that the open interval (a, b) is an incomplete subspace of R.

Ex.13. Let X be the set of all ordered $n-$tuples $x = (\xi_{1,} \xi_{2,,,,,,,,,,,,} \xi_n)$ of real numbers and

$$d(x, y) = \max_j |\xi_j - \eta_j|$$

where $y = (\eta_j)$. Show that (X, d) is complete.

Ex.14. Let $M \subset \ell^\infty$ be the subspace consisting of all sequence $x = (\xi_j)$ with at most finitely many nonzero terms .Find a Cauchy sequence in M which does not converge in M ,so that M is not complete.

Ex.15.Show that the set X of all integers with metric d defined by $d(m, n) = |m - n|$ is a complete metric space.

Ex.16.Let X be the set of all positive integers and $d(m, n) = |m^{-1} - n^{-1}|$. Show that (X, d) is not complete.

Ex.17. Show that a discrete metric space is complete.

Ex.18.Let X be metric space of all real sequences $x = (\xi_j)$ each of which has only finitely Nonzero terms, and $d(x, y) = \Sigma|\xi_j - \eta_j|$, when $y = (\eta_j)$. Show that $(x_n), x_n = (\xi_j^{(n)})$, $\xi_j^{(n)} = j^{-2}$ for $j = 1, 2.., n$ and $\xi_j^{(n)} = 0$ for $j > n$ is Cauchy but does not converge.

Ex.19. Show that,by given a example ,that a complete and an incomplete metric spaces may be Homeomorphic.

Ex.20. If (X, d) is complete, show that $(X, \bar{d})$,where $\bar{d} = \dfrac{d}{1+d}$ is complete.

<u>HINTS</u> (Problems based on Module –I)

Hint.1: Use definition . Ans (i) k>0 (ii) k=0

Hint.2: Use Triangle inequality $d(x,y) \le d(x,z)+d(z,w)+d(w,y)$

Similarity, $d(z,w) \le d(z,x)+d(x,y)+d(y,w)$

Hint.4: Choose $x=(x_k)\, where\, x_k = \frac{1}{k}$

Hint.5: No. Because D(A,B)=0 $\ne>$ A=B e.g. Choose $\begin{matrix} A=\{a,a_{1,}a_2.......\} \\ B=\{a,b_{1,}b_2.......\} \end{matrix}$ where $a_i \ne b_i$

clearly $A \ne B$ but D=(A,B)=0 & $A \cap B \ne \phi$.

Hint.6: Yes. ; Hint.7: Yes.

Hint.8: Any subset is open since for any $a \in A,$ the open ball $B\left(a,\frac{1}{2}\right)=\{a\} \subset A.$

Similarly A^c is open, so that $(A^c)^c$=A is closed.

Hint.9: use Definition.

Ans (a) The integer, (b) R, (c) $\mathbb{C}$, (d) $\{z:|z| \le 1\}$.

Hint.10:Let X be separable .So it has a countable dense subset Y i.e. $\bar{Y}=X$ .Let $x \in X$ & $\in>0$ be given.Since Y is dense in X and $x \in \bar{Y}$, so that the $\in$ neibourhood $B(x;\in)$ of x contains a $y \in Y$, and $d(x,y)<\in$.Conversely, if X has a countable subset Y with the property given in the problem, every $x \in X$ is a point of Y or an accumulation point of Y. Hence $x \in Y$, s result follows .

Hint.11:Since $|a_n - a_m| = |d(x_n,y_n)-d(x_m,y_m)|$

$\le d(x_n,x_m)+d(y_n,y_m) \to o$ as $n \to \infty$ which shows that (a_n) is a Cauchy sequence of real numbers . Hence convergent.

Hint.12:Choose $(a_n)=\left(a+\frac{1}{n}\right)$ which is a Cauchy sequence in (a,b) but does not converge.

Hint.14:Choose (x_n), where $x_n=\left(1,\frac{1}{2},\frac{1}{3},...,\frac{1}{n},0,0...\right)$ which is Cauchy in M but does not converge.

Hint.15:Consider a sequence $x \equiv x \equiv (x_k)=(\alpha_1,\alpha_2..,\alpha_{ni},\alpha,\alpha...)$

Where $x_k = \alpha$ for $k \ge n, \alpha$ is an integer.This is a Cauchy and converges to $\alpha \in X$.

Hint.16: Choose (x_n) when $x_n = n$ which is Cauchy but does not converge.

Hint.17: Constant sequence are Cauchy and convergent.

Hint.18: For every $\in > 0$,there is an N $s.t.$ for $n > m > N$,

$$d(x_n, x_m) = \sum_{j=m+1}^{n} \frac{1}{j^2} < \in .$$

But (x_n) does not converge to any $x = (\xi_j) \in X$

Because $\xi_j = 0$ for $j > \bar{N}$ so that for $n > \bar{N}$,

$$d(x_n, x) = |1 - \xi_1| + \left|\frac{1}{4} - \xi_2\right| + \ldots + \frac{1}{(\bar{N}+1)^2} + \ldots + \frac{1}{n^2} > \frac{1}{(\bar{N}+1)^2}$$

And

$d(x_n, x)$ $\to 0$ as $n \to \infty$ is imposible .

Hint.19: (Def) A homeomorphism is a continuous bijective mapping.

$T : X \to Y$ whose inverse is continuous; the metric space X and Y are then said to be homeomorphic. $e.g.$ A mapping $T : R \to (-1,1)$ defined as $Tx = \frac{2}{\pi}\tan^{-1} x$ with metric $d(x, y) = |x - y|$ on R .Clearly T is 1-1,into & bi continuous so $R \cong (-1,1)$

But R is complete while $(-1,1)$ is an incomplete metric space.

Hint.20: $\bar{d} = (x_m, x_n) < \in < \frac{1}{2}$ then

$$d(x_m, x_n) = \frac{\bar{d}(x_m, x_n)}{1 - \bar{d}(x_m, x_n)} < 2\,d(x_m, x_n).$$

Hence if (x_n) is Cauchy is $\left(X, \bar{d}\right)$, it is Cauchy in (X, d), and its limit in $\left(X, \bar{d}\right)$.

Problems on Module-II (Normed and Banach Spaces)

Ex.-1. Let (X, $\| \ \|_i$), i= 1, 2, ∞ be normed spaces of all ordered pairs

x=$(\xi_1, \xi_2), y = (\eta_1, \eta_2)$,..... of real numbers where

$\| \ \|_i$, i= 1,2,∞ are defined as

$\|x\|_1 = |\xi_1| + |\xi_2|$; $\|x\|_2 = \left(\xi_1^2 + \xi_2^2\right)^{1/2}$; $\|x\|_\infty = \max\left(|\xi_1|, |\xi_2|\right)$

How does unit sphere in these norms look like?

Ex.-2. Show that the discrete metric on a vector space X$\neq \{0\}$ can not be obtained from a norm.

Ex.-3. In ℓ^∞,let Υ be the subset of all sequences with only finitely many non zero terms. Show that Υ is a subspace of ℓ^∞ but not a closed subspace.

Ex.-4. Give examples of subspaces of ℓ^∞ and ℓ^2 which are not closed.

Ex.-5. Show that $\Re^n$ and $\mathbb{C}^n$ are not compact.

Ex.-6. Show that a discrete metric space X consisting of infinitely many points is not compact.

Ex.-7. Give examples of compact and non compact curves in the plane $\Re^2$.

Ex.-8. Show that $\Re$ and $\mathbb{C}$ are locally compact.

Ex.-9. Let X and Y be metric spaces. X is compact and $T : X \to Y$ bijective and continuous. Show that T is homeomorphism.

Ex.-10. Show that the operators $T_1, T_2 ..., T_4$ from R^2 into R^2 defined by

$(\xi_1, \xi_2) \to (\xi_1, 0), \to (0, \xi_2), \to (\xi_2, \xi_1)$ and $\to (\sqrt{\xi_1}, \sqrt{\xi_2})$

respectively, are linear.

Ex.-11. What are the domain, range and null space of T_1, T_2, T_3 in exercise 9.

Ex.-12. Let $T : X \to Y$ be a linear operator. Show that the image of a subspace V of X is a vector space, and so is the inverse image of a subspace W of X.

Ex.-13. Let X be the vector space of all complex 2×2 matrices and define $T : X \to X$ by Tx=bx, where $b \in X$ is fixed and bx denotes the usual product of matrices. Show that T is linear. Under what condition does T^{-1} exist?

Ex.-14. Let $T : D(T) \to Y$ be a linear operator whose inverse exists. If $\{x_1, x_2, ..., x_n\}$ is a Linearly Independant set in D(T), show that the set $\{Tx_1, Tx_2, ...Tx_n\}$ is L.I.

Ex.-15. Let $T : X \to Y$ be a linear operator and dim X = dim Y = n $< \infty$. Show that $R(T) = Y \Leftrightarrow T^{-1}$ exists.

Ex.-16. Consider the vector space X of all real-valued functions which are defined on R and have derivatives of all orders everywhere on R. Define $T : X \to X$ by y(t)=Tx(t)=x'(t),show that R(T) is all of X but T^{-1} does not exist.

Ex.-17. Let X and Y be normed spaces. Show that a linear operator $T : X \to Y$ is bounded if and only if T maps bounded sets in X into bounded set in Y.

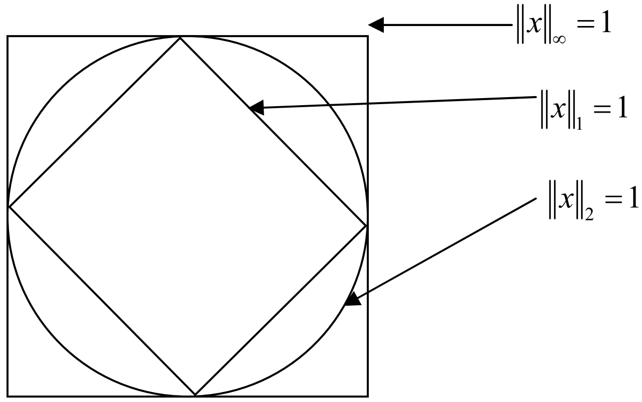
Ex.-18. If $T \neq 0$ is a bounded linear operator, show that for any $x \in D(T)$ s.t. $\|x\| < 1$ we have the strict inequality $\|Tx\| < \|T\|$.

Ex.-19. Show that the functional defined on C[a , b] by $f_1(x) = \int_a^b x(t) y_0(t) dt; f_2(x) = x(a) + \beta x(b)$,where $y_0 \in C[a,b], \alpha, \beta$ fixed are linear and bounded.

Ex.-20. Find the norm of the linear functional f defined on C[-1,1] by $f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt$.

HINTS (Problems on Module-II)

Hint.1:



Hint.2:

$$\because d(x,y) = \begin{cases} 0 & if \quad x = y \\ 1 & if \quad x \neq y \end{cases} \quad \text{, where} \quad d(\alpha x, \alpha y) \neq |\alpha| d(x,y)$$

Hint.3: Let $x^{(n)} = (1, \frac{1}{2}, \frac{1}{3}, ...1/n, 0, 0.....) = \left(x_j^{(n)}\right)$ where $x_j^{(n)}$ has 0 value after j>n.

Clearly $x^{(\eta)} \in \ell^\infty$ as well as $x^{(\eta)} \in \Upsilon$ but $\lim_{n\to\infty} x^{(\eta)} \notin \Upsilon$.

Hint.4: Let Υ be the subset of all sequences with only finitely many non zero terms.

e.g. $\Upsilon = \{x^{(n)} = (1, \frac{1}{2}, \frac{1}{3}, ...\frac{1}{n}, 0, ...), n = 1, 2, ...\} \subset \ell^\infty \;\&\subset \ell^2$ but not closed.

Hint.6: By def. of Discrete metric, any sequences (x_n) cannot have convergent subsequence as $d(x_i, x_j) = 1 \text{ if } i \neq j$.

Hint.7: As $\Re^2$ is of finite dimension, So every closed & bounded set is compact.

Choose $X = \{(x,y) = a_1 \leq x \leq b_{1,} a_2 \leq x \leq b_1\}$ which is compact

But $\{(x,y) = a_1 < x < b_1, a_1 < y < b_2\}$ is not compact.

Hint.8: (def.) A metric space X is said to be locally compact if every point of X has a compact neighbourhood. Result follows (obviously).

Hint.9: Only to show T^{-1} is continuous i.e. Inverse image of open set under T^{-1} is open.

OR. If $\gamma_n \to \gamma$. Then $T^{-1}(\gamma_n) \to T^{-1}(\gamma)$. It will follow from the fact that X is compact.

Hint.11: The domain is $\Re^2$.The ranges are the ξ_1-axis, the ξ_2-axis ,$\Re^2$.The null spaces are the ξ_2-axis, the ξ_1-axis, the origin.

Hint.12. Let $Tx_1, Tx_2 \in T(V)$.Then $x_1, x_2 \in V, \alpha x_1 + \beta x_2 \in V$. Hence $T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 \in T(V)$. Let x_1, x_2 be in that inverse image. Then $Tx_1, Tx_2 \in W$, $\alpha Tx_1 + \beta Tx_2 \in W$, $\alpha Tx_1 + \beta Tx_2 = T(\alpha x_1 + \beta x_2)$, so that $\alpha x_1 + \beta x_2$ is an element of that inverse image.

Hint.13. $|b| \neq 0$

Hint. 14. If $\{Tx_1, Tx_2, ...Tx_n\}$ is not L.I. then $\ni$ some $\alpha_i \neq 0$ $\alpha_1 Tx_1 + ... + \alpha_i Tx_i + .. + \alpha_n Tx_n = 0$. Since T^{-1} exists and linear , $T^{-1}\left(\alpha_1 Tx_1 + ... + \alpha_n Tx_n\right) = \alpha_1 x_1 + ... + \alpha_n x_n = 0$ when $\alpha_i \neq 0$ which shows $\{x_1, x_2, ..., x_n\}$ is L.D., a contradiction.

Hint.16: R (T) =X since for every $y \in X$ we have y=Tx, where $x(t) = \int_0^t y(\tau) d\tau$. But n T^{-1} does not exist since Tx=0 for every constant function.

Hint.17: Apply definition of bounded operator.

Hint.18: Since $\|Tx\| = \|T\|.\|x\| < \|T\|$ as $\|x\| < 1$.

Hint.20: $|f(x)| \leq 2\|x\|, \therefore \|f\| \leq 2$. For converse , choose $x(t) = -1$ on $[-1,1]$. So $\|x\| = 1$

$$\|f\| \geq \left| -\int_{-1}^{0} dt + \int_{0}^{1} dt \right| = 2 \quad \therefore \|f\| = 2.$$

Problems on Module III (IPS/Hilbert space)

Ex.-1. If $x \perp y$ in an IPS X,Show that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Ex.-2. If X in exercise 1 is a real vector space, show that ,conversely, the given relation implies that $x \perp y$. Show that this may not hold if X is complex. Give examples.

Ex,-3. If an IPS X is real vector space, show that the condition $\|x\| = \|y\|$ implies $< x+y, x-y >= 0$ .What does this mean geometrically if X=R^2 ?

Ex.-4. (Apollonius identity): For any elements x, y, z in an IPS X, show that

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{1}{2}(x+y)\right\|^2.$$

Ex.-5. Let $x \neq 0$ and $y \neq 0$. If $x \perp y$,show that {x,y} is a Linearly Independent set.

Ex.-6. If in an IPS X, $< x,u >=< x,v >$ for all x, show that u=v.

Ex.-7. Let X be the vector space of all ordered pairs of complex numbers. Can we obtain the norm defined on X by $\|x\| = |\xi_1| + |\xi_2|, x = (\xi_1, \xi_2) \in X$ from an Inner product?

Ex.-8. If X is a finite dimensional vector space and (e_j) is a basis for X, show that an inner product on X is completely determined by its values $\gamma_{jk} =< e_j, e_k >$. Can we choose scalars γ_{jk} in a completely arbitrary fashion?

Ex.-9. Show that for a sequence (x_n) in an IPS X , the conditions $\|x_n\| \to \|x\|$ and $< x_n, x >\to< x,x >$ imply convergence $x_n \to x$.

Ex.-10. Show that in an IPS X,
$x \perp y \Leftrightarrow$ we have $\|x+\alpha y\| = \|x-\alpha y\|$ for all scalars α.

Ex.-11. Show that in an IPS X, $x \perp y \Leftrightarrow \|x+\alpha y\| \geq \|x\|$ for all scalars.

Ex.-12. Let V be the vector space of all continuous complex valued functions on J=[a, b]. Let $X_1 = (V, \|...\|_\infty)$,where $\|x\|_\infty = \max_{t\in J}|x(t)|$; and let $X_2 = (V, \|...\|_2)$,where $\|x\|_2 =< x,x >^{\frac{1}{2}}, < x,y >= \int_a^b x(t)\overline{y(t)}dt$. Show that the identity mapping $x \mapsto x$ of X_1 onto X_2 is continuous. Is it Homeomorphism?

Ex.-13. Let H be a Hilbert space, $M \subset H$ a convex subset, and (x_n) is a sequence in M such that $\|x_n\| \to d$, where $d = \inf_{x \in M} \|x\|$. Show that (x_n) converges in H.

Ex.-14. If (e_k) is an orthonormal sequence in an IPS X, and $x \in X$, show that x-y with y given by $y = \sum_{1}^{n} \alpha_k e_k , \alpha_k = < x, e_k >$ is orthogonal to the subspace $Y_n = span\{e_1, e_2, ..., e_n\}$.

Ex.-15. Let (e_k) be any orthonormal sequence in an IPS X. Show that for any $x, y \in X$,

$$\sum_{k=1}^{\infty} |< x, e_k >< y, e_k >| \leq \|x\| \|y\| .$$

Ex.-16. Show that in a Hilbert Space H,convergence of $\sum \|x_j\|$ implies convergence of $\sum x_j$

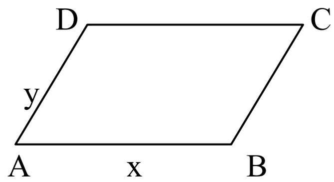
Hints on Problems on Module III

Hint.1: Use $\|x\|^2 = <x,x>$ and the fact that $<x,y> = 0$, if $x \perp y$.

Hint. 2 : By Assumption,

$$0 = <x+y, x+y> - \|x\|^2 - \|y\|^2 = <x,y> + \overline{<x,y>} = 2\,\mathrm{Re} <x,y>.$$

Hint.3: Start $<x+y, x-y> = <x,x> + <y,-y> = \|x\|^2 - \|y\|^2 = 0$ as X is real.
Geometrically: If x & y are the vectors representing the sides of a parallelogram, then x+y and x-y will represent the diagonal which are $\perp$.



Hint 4: Use $\|x\|^2 = <x,x>$ OR use parallelogram equality.

Hint.5: Suppose $\alpha_1 x + \alpha_2 y = 0$ where α_1, α_2 are scalars. Consider

$<\alpha_1 x + \alpha_2 y, x> = <0, x>$

$\Rightarrow \alpha_1 \|x\|^2 = 0$ as $<x,y> = 0$.

$\Rightarrow \alpha_1 = 0$ as $\|x\| \neq 0$.Similarly, one can show that $\alpha_2 = 0$. So {x, y} is L.I.set.

Hint.6 : Given $<x, u-v> = 0$ .Choose x=u-v.

$\Rightarrow \|u-v\|^2 = 0 \Rightarrow u = v.$

Hint. 7: No. because the vectors x= (1,1), y=(1,-1) do not satisfy parallelogram equality.

Hint.8: Use x=$\sum_{1}^{n} \alpha_i e_i$ & $y = \sum_{j=1}^{n} \alpha_j e_j$ .Consider $<x,y> = <\sum_{i=1}^{n} \alpha_i e_{i,} \sum_{j=1}^{n} \alpha_j e_j>$.

Open it so we get that it depends on $\gamma_{jk} = <e_j, e_k>$

II Part: Answer:- NO. Because $\gamma_{jk} = <e_j, e_k> = \overline{<e_k, e_j>} = \overline{\gamma_{kj}}$.

Hint.9 : We have

$\|x_n - x\|^2 = <x_n - x, x_n - x>$

$= \|x_n\|^2 - <x_n, x> - <x, x_n> + \|x\|^2$

$\Rightarrow 2\|x\|^2 - 2<x,x> = 0$ as $n \to \infty$.

Hint.10 : From

$< x \pm \alpha y, x \pm \alpha y > = \|x\|^2 \pm \bar{\alpha} < x, y > \pm \alpha < y, x > + |\alpha|^2 \|y\|^2$ condition follows as $x \perp y$.

Conversely, $\|x + \alpha y\| = \|x - \alpha y\|$

$$\Rightarrow \bar{\alpha} < x, y > + \alpha < y, x > = 0.$$

Choose $\alpha = 1$ if the space is real which implies $x \perp y$.

Choose $\alpha = 1, \alpha = i$, if the space is complex then we get $< x, y > = 0 \Rightarrow x \perp y$.

Hint.11 : Follows from the hint given in Ex.-10.

Hint.12 : Since

$$\|x\|_2^2 = \int_a^b |x(t)|^2 \, dt \leq (b-a)\|x\|_\infty^2 \text{ -------(A)}$$

Suppose $x_n \to 0$ in X_1 i.e. $\|x_n\|_\infty \to 0$ as $n \to \infty$.

So by (A) , $x_n \xrightarrow[will]{X_2} 0$.

Hence I is continuous.

Part-II: Answer No. because X_2 is not complete.

Hint.13 : (x_n) is Cauchy, since from the assumption and the parallelogram equality, we have,

$$\|x_n - x_m\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_n + x_m\|^2$$

$\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4d^2$ (since M is convex so

$\frac{x_n + x_m}{2} \in M \ \therefore \left\|\frac{x_n + x_m}{2}\right\|^2 \geq d^2 \because \inf\|x_n\| = d, x_n \in M$).

Hint.14 : $y \in Y_n, x = y + (x - y),$ and $x - y \perp e_m$,

Since $< x - y, e_m > = < x - \sum \alpha_k e_k, e_m >$

$= < x, e_m > - \alpha_m = 0$.

Hint.15: Use Cauchy Schwaz's Inequality & Bessel's Inequality, we get

$$\sum |< x, e_k >< y, e_k >| = (\sum |< x_1 e_k >|^2)^{\frac{1}{2}} (\sum |< y, e_k >|^2)^{\frac{1}{2}} \leq \|x\|\|y\|.$$

Hint.16 : Let $\delta_n = x_1 + x_2 + ... + x_n$

$\|\delta_n - \delta_m\| \leq \sum_{j=m}^{n} \|x_j\| \leq \sum_{j=m}^{\infty} \|x_j\| \to 0$ as $m \to \infty$ .So

(δ_n) is a Cauchy. Since H is complete, hence (δ_n) will converge .

$\therefore \sum x_j$ converge in H.

Problems On Module <u>IV</u> (On Fundamental theorems)

Ex.1. Let $f_n : \ell^1 \to R$ be a sequence of bounded linear functionals defined as $f_n(x) = \xi_n$ where $x = (\xi_n) \in \ell^1$. show that (f_n) converge strongly to 0 but not uniformly.

Ex.2. Let $T_n \in B(X,Y)$ where X is a Banach space and Y a normed space. If (T_n) is strongly convergent with limit T, then $T \in B(X,Y)$.

Ex.3. If $x_n \in C[a,b]$ and $x_n \xrightarrow{\omega} x \in C[a,b]$. Show that (x_n) is point wise convergent on $[a,b]$.

Ex.4. If $x_n \xrightarrow{\omega} x_o$ in a normed space X. Show that $x_o \in \bar{Y}$, Where Y= span (x_n).

Ex.5. Let $T_n = S^n$, where the operator $S : \ell^2 \to \ell^2$ is defined by $S\{(\xi_n, \xi_2, \xi_3 ...)\} = (\xi_3, \xi_4 ...)$.

Find a bound for $\|T_n x\|$; $\lim_{n\to\infty} \|T_n x\|$, $\|T_n\|$ *and* $\lim_{n\to\infty} \|T_n\|$.

Ex.6. Let X be a Banach space, Y a normed space and $T_n \in B(X,Y)$ such that $(T_n x)$ is Cauchy in Y for every $x \in X$. show that $(\|T_n\|)$ is bounded.

Ex.7. If (x_n) in a Banach space X is such that $(f(x_n))$ is bounded for all $f \in X'$.
Show that $(\|x_n\|)$ is bounded.

Ex.8. If a normed space X is reflexive, Show that X' is reflexive.

Ex.9. If x_o in a normed space X is such that $|f(x_o)| \le c$ for all $f \in X'$ of norm 1 .show that $\|x_o\| \le c$.

Ex.10. Let Y be a closed sub space of a normed space X such that every $f \in X'$ which is zero every where on Y is zero every where on the whole space X. Show that $Y = X$

Ex.11. Prove that $(S+T)^\times = S^\times + T^\times ; (\alpha T)^\times ; (\alpha T)^\times = \alpha T^\times$
Where $T^\times$ is the adjoint operator of T.

Ex.12. Prove $(ST)^\times = T^\times S^\times$

Ex.13. Show that $(T^n)^{\times} = (T^{\times})^n$.

Ex.14. Of what category is the set of all rational number (a) in $\mathbb{R}$, (b) in itself, (Taken usual metric).

Ex.15. Find all rare sets in a discrete metric space X.

Ex.16. Show that a subset M of a metric space X is rare in X if and only if is $\left(\bar{M}\right)^c$ is dense in X.

Ex.17. Show that completeness of X is essential in uniform bounded ness theorem and cannot be omitted.

Hints on Problems On Module IV

Hint.1 : Since $x \in \ell^1 \Rightarrow \sum_1^\infty |\xi_n| < \infty \Rightarrow |\xi_n| \to 0$ as $n \to \infty$.

ie $f_n(x) \to 0$ as $n \to \infty$ but $\|f_n\| = 1 \nrightarrow 0$.

Hint.2 T linear follows

$$\lim_{n\to\infty} T_n(\alpha x + py) = \lim_{n\to\infty}\{(\alpha T_n x) + (\beta T_n x)\} \Rightarrow T(\alpha x + \beta y) = \alpha Tx + \beta Ty .$$

T is bounded :- Since $T_n \xrightarrow{s} T$ i.e. $\|(T_n - T)x\| \to 0$ for all $x \in X$.

So $(T_n x)$ is bounded for every x. Since X is complete, so $(\|T_n\|)$ is bounded by uniform bounded ness theorem. Hence

$\|T_n x\| \leq \|T_n\|\|x\| \leq M\|x\|$. Taking limit $\Rightarrow T$ is bounded.

Hint .3 : A bounded linear functional on $C[a,b]$ is δ_{t_0} defined by $\delta_{t_o}(x) = x(t_o)$, when $t_o \in [a,b]$.

Given $x_n \xrightarrow{\omega} x \Rightarrow |\delta_{t_o}(x_n) - \delta_{t_o}(x)| \to 0$ as $n \to \infty \Rightarrow x_n(t_o) \to x(t_o)$ as $n \to \infty$.

Hint.4 : Use Lemma:- ''Let Y be a proper closed sub-space of a normed space X and let $x_o \in X - Y$ be arbitrary point and $\delta = \inf_{\bar{y} \in Y} \left\| \bar{y} - x_o \right\| > 0$, then there exists an $\bar{f} \in X'$, dual of X such that

$\left\| \bar{f} \right\| = 1, \bar{f}(y) = 0$ for all $y \in Y$ and $\bar{f}(x_o) = \delta$.''

suppose $x_o \notin \bar{Y}$ which is a closed sub space of X. so by the above result ,

for $x \in X - Y$, $\delta = \inf_{\bar{y} \in Y} \left\| \bar{y} - x_o \right\| > 0$,hence there exists $\bar{f} \in X'$ s.t. $\left\| \tilde{f} \right\| = 1$ & $\bar{f}(x_n) = 0$

for $x_n \in \bar{Y}$. Also $\bar{f}(x_o) = \delta$. So $\bar{f}(x_n) \not\to f(x_o)$ which is a contradiction that $x_n \xrightarrow{\omega} x_o$.

Hint.5 : $T_n = S^n$. $T_n(x) = (\xi_{2n+1}, \xi_{2n+2}, ...)$

(i) $\|T_n x\|^2 = \sum_{k=2n+1}^{\infty} |\xi_k|^2 \leq \sum_{k=1}^{\infty} |\xi_k|^2 = \|x\|^2 \Rightarrow \|T_n x\| \leq \|x\|$.

(ii) $\lim_{n\to\infty} \|T_n x\| = 0$.

(iii) $\|T_n\| \leq 1$ as $\|T_n x\| \leq \|x\|$. For converse, choose $x = \left(0, 0, .., \underset{(2n+1)place}{1}, 0....\right)$ so $\|T_n\| \geq 1$.

$\therefore \|T_n\| = 1$.

(v) $\lim_{n\to\infty} \|T_n\| = 1$.

Hint. 6 : Since $(T_n x)$ is Cauchy in Y for every x, so it is bounded for each $x \in X$.
Hence by uniform bounded ness theorem, $(\|T_n\|)$ is bounded.

Hint.7 : Suppose $f(x_n) = g_n(f)$. Then $\{g_n(f)\}$ is bounded for every $f \in X'$. So by uniform bounded ness theorem $(\|g_n\|)$ is bounded and $\|x_n\| = \|g_n\|$.

Hint. 8 : Let $h \in X'''$. For every $g \in X''$ there is an $x \in X$ such that $g = Cx$ since X is reflexive. Hence $h(g) = h(Cx) = f(x)$ defines a bounded linear functional f on X and $C_1 f = h$, where $C_1 : X' \to X'''$ is the canonical mapping. Hence C_1 is surjective, so that X' is reflexive.

Hint. 9 : suppose $\|x_o\| > c$. Then by Lemma: Let X be a normed space and let $x_o \neq 0$ be any element of X. Then there exist a bounded linear functional $\tilde{f}$ on X s.t. $\|\tilde{f}\| = 1$ & $\tilde{f}(x_o) = \|x_o\|$.

$\|x_o\| > c$ would imply the existence of an $\tilde{f} \in X'$ s.t. $\|\tilde{f}\| = 1$ and $\tilde{f}(x_o) = \|x_o\| > c.$

Hint. 10 : If $Y \neq X$, there is an $x_o \in X - Y$, and $\delta = \inf_{y \in Y} \|y - x_o\| > 0$ since Y is closed.

Use the Lemma as given in Ex 4 (Hint) .

By this Lemma , there is on $\tilde{f} \in X'$ which is zero on Y but not zero at x_o, which contradicts our assumption.

Hint. 11 : $\left((S+T)^{\times} g\right)(x) = g\left((S+T)x\right) = g(Sx) + g(Tx) = \left(S^{\times} g\right)(x) + \left(T^{\times} g\right)(x)$. Similarly others.

Hint. 12 : $\left((ST)^{\times} g\right)(x) = g(STx) = (S^{\times} g)(Tx) = \left(T^{\times}\left(S^{\times} g\right)\right)(x) = \left(T^{\times} S^{\times} g\right)(x).$

Hint. 13 : Follows from Induction.

Hint 14 : (a) first (b) first.

Hint.15 : ϕ, because every subset of X is open.

Hint. 16 : The closure of $\left(\bar{M}\right)^c$ is all of X if and if $\bar{M}$ has no interior points, So that every $x \in \bar{M}$ is a point of accumulation of $\left(\bar{M}\right)^c$.

Hint.17 : Consider the sub space $X \subset \ell^{\infty}$ consisting of all $x = (\xi_j)$ s.t. $\xi_j = 0$ for $j \geq J \in \mathbb{N}$, where J depends on x, and let T_n be defined by $T_n x = f_n(x) = n \ \xi_n$.

Clearly $(\|T_n X\|)$ *is bounded* $\forall x$ but $\|T_n\|$ is not bounded.